

# Cohomology of Local Systems on $X_\Gamma$

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## 1 Local Systems

**Definition 1.1.** Let  $X$  be a topological space and let  $S$  be a set (usually with additional structure, ring module, etc). The constant sheaf  $S_X$  is defined to be

$$S_X(U) = \{f : U \rightarrow S \mid f \text{ is continuous and } S \text{ has the discrete topology}\}$$

**Remark.** Equivalently,  $S_X$  is the sheaf whose sections are locally constant functions  $f : U \rightarrow S$  and also is equivalent to the sheafification of the constant presheaf which assigns  $A$  to every open set.

**Remark.** When  $U$  is connected,  $S_X(U) = S$ .

**Definition 1.2.** Let  $A$  be a ring. Then an  $A$ -local system on a topological space  $X$  is a sheaf  $\mathcal{L} \in \text{mod}(A_X)$  s.t. there exists a covering of  $X$  by  $\{U_i\}$  s.t.  $\mathcal{L}|_{U_i} = \underline{M}_i$  where  $\underline{M}_i$  is the constant sheaf associated to the  $R$ -module  $M_i$ . In other words, a local system is the same thing as a locally constant sheaf.

**Remark.** If  $X$  is connected, then all the  $M_i$  are the same.

**Example 1.**  $A_X$  is an  $A$ -local system.

**Example 2.** Let  $D$  be an open connected subset of  $\mathbb{C}$ . Then the sheaf  $\mathcal{F}$  of solutions to LODE, namely

$$\mathcal{F}(U) = \left\{ f : U \rightarrow \mathbb{C} \mid f^{(n)} + a_1(z)f^{(n-1)} + \dots + a_n(z) = 0 \right\}$$

where  $a_i(z)$  are holomorphic forms a  $\mathbb{C}$ -local system. Existence and uniqueness of solutions of ODE on simply connected regions means that by choosing a disc  $D(z)$  around each point  $z \in D$ , we see that the initial conditions  $f^{(k)} = y_k$  give an isomorphism

$$\mathcal{F}|_{D(z)} \cong \underline{\mathbb{C}^n}$$

**Example 3.** We can generalize the above example to differential equations with "singularities" if we have a flat connection. Let  $E \rightarrow X$  be a vector bundle with a flat connection  $\nabla$ , then  $E^\nabla$  will be a local system where

$$E^\nabla(U) = \{\text{sections } s \in \Gamma(U, E) \text{ which are horizontal: } \nabla s = 0\}$$

Don't think too much about the previous example. For our purposes  $A$  will be a field where  $A = \mathbb{R}, \mathbb{C}$ . Then  $A$ -local systems are sort of like vector bundles in that the stalks will be vector spaces. But they aren't the same thing because local systems are "discrete." However, local systems are essentially vector bundles with a flat connection which is an incarnation of the Riemann-Hilbert correspondence.

## 2 Monodromy

Sections of a vector bundle naturally form a sheaf and in fact, all sheaves can be thought of from this point of view. That is for any sheaf  $\mathcal{F}$  on a topological space  $X$  there is a topological space  $\text{Et}(\mathcal{F})$ , called the étalé space of  $\mathcal{F}$ , along with a projection map to  $X$  such that the sections of the projection over an open set  $U$  will be the sections of  $\mathcal{F}(U)$ . The construction of  $\text{Et}(\mathcal{F})$  proceeds as follows. As a set it will be the disjoint union of the stalks

$$\text{Et}(\mathcal{F}) = \bigsqcup_{x \in X} \mathcal{F}_x$$

A basis for the topology on  $\text{Et}(\mathcal{F})$  will be generated by sets of the form

$$U_s = \{(x, s_x) \mid x \in U, s \in \mathcal{F}(U), U \text{ is open in } X\}$$

aka a certain set of elements in the stalks over each point in  $U$ , and we have exactly one element from each stalk. Thus, under the subspace topology, each fiber will have the discrete topology. You can check that under the projection  $\pi : \text{Et}(\mathcal{F}) \rightarrow X$ , sections of  $\pi$  over  $U$  return  $\mathcal{F}(U)$  (specifically they will be sections of the sheafification of  $\mathcal{F}$ .)

**Lemma 2.1.** *The étalé space of the constant sheaf  $A_X$  on a locally connected space  $X$  will be  $X \times A$  where  $A$  has the discrete topology.*

*Proof.* As a set we have that  $\text{Et}(A_X) = X \times A$  as the stalks of  $A_X$  are all  $A$  so it suffices to show the étalé topology coincides with the product topology. Since  $X$  is locally connected, any basic open  $U_s$  in the étalé topology is the union of  $\{U_{i,s_i}\}$  where  $U_i$  connected and  $s_i = \text{rest}_{U,U_i}(s)$ . As  $A_X$  is a constant sheaf,  $A_X(U) = A$ , so  $s_i = a$  is constant. Then the open set  $U_{i,s_i}$  in  $\text{Et}(A_X)$  will be of the form  $U_i \times a_i$  which is open in the product topology as  $A$  has the discrete topology. Thus  $U_s$  is the union of opens in the product topology and thus is also open in the product topology. To go the other way, note that every open  $U \times C$  in the product topology can be written as the union of opens  $U_i \times a_j$  where  $\cup U_i = U, a_j \in C, U_i$  is connected since  $A$  has the discrete topology. But  $U_i \times a_j$  is an open set in the étalé topology because  $U_i$  is connected and thus  $U \times C$  being the union of all these guys must also be open in the étalé topology as desired.

**Lemma 2.2.** *Any  $A$ -local system  $\mathcal{L}$  on a connected, simply connected, and locally connected space  $X$  is a constant sheaf  $\underline{M}$  for some  $A$ -module  $M$ .*

*Proof.* We claim that the étalé space of a locally constant sheaf will be a covering space for  $X$ . We know that there exists a covering  $\{U_i\}$  of  $X$  for which  $\mathcal{L}|_{U_i} \cong \underline{M}$ , but by the previous lemma the étalé space of the constant sheaf  $\underline{M}$  will be the product space  $X \times M$ . In other words,  $\pi^{-1}(U_i) \cong U_i \times M$  and so we see  $\pi : \text{Et}(\mathcal{L}) \rightarrow X$  will be a fiber bundle. But since each fiber  $A_i$  is discrete, it follows that  $\text{Et}(\mathcal{L})$  is actually a covering space for  $X$

For  $X$  simply connected,  $\pi_1(X) = \{e\}$ , but covering spaces correspond to subgroups of  $\pi_1(X)$  by the classification of covering spaces, so  $\text{Et}(\mathcal{L})$  must be a trivial covering of  $X$  and thus is of the form  $X \times A$ . By construction taking the sheaf of sections on  $\text{Et}(\mathcal{L})$  will return  $\mathcal{L}$  and the sheaf of sections of  $X \times M$  will precisely be  $\underline{M}$  so  $\mathcal{L}$  is a constant sheaf.

**Remark.** As  $S^2$  is simply connected, locally path connected, any  $\mathbb{C}$  local system is constant(trivial) on  $S^2$ . However, the same isn't true for vector bundles. Complex line bundles on  $S^2$  up to isomorphism are in bijection with  $\pi_1(\mathrm{GL}_1(\mathbb{C})) = \pi_1(\mathbb{C}^\times) = \mathbb{Z}$  by the clutching construction. Or if you like algebraic geometry, then  $S^2 = \mathbb{P}^1(\mathbb{C})$  and line bundles on  $\mathbb{P}^1$  are exactly given by  $\mathcal{O}_X(n)$  for  $n \in \mathbb{Z}$ . Thus local systems are "simpler" in the sense that you only need the first homotopy group to vanish to be trivial while vector bundles need all homotopy groups to vanish.

**Theorem 1**

Assume  $X$  is connected, locally connected, path-connected, paracompact, Hausdorff, etc. Then the following categories are equivalent.

- (i)  $A$ -local systems on  $X$ .
- (ii) Covariant functors  $L$  from the fundamental groupoid of  $X$  to the category of  $A$ -modules.
- (iii) Representations  $\rho : \pi_1(X, x_0) \rightarrow \mathrm{Aut}_A(M)$ ,  $M$  is an  $A$ -module.

*Proof.* We describe how a local system  $\mathcal{L}$  gives maps between fibers/stalks. Consider a path  $\gamma : I \rightarrow X$  between  $x_0$  and  $x_1$ , then note that  $\gamma^{-1}\mathcal{L}$  will be a local system on  $[0, 1]$ . This follows from the étalé space point of view as the pullback sheaf will be the sheaf of sections of the pullback bundle and locally constant sheafs locally are products so we have the fiber square

$$\begin{array}{ccc} \gamma^{-1}(U) \times A & \longrightarrow & U \times A \\ \downarrow & & \downarrow \\ \gamma^{-1}(U) & \longrightarrow & U \end{array}$$

So we see that  $\gamma^{-1}(U)$  will give us a trivializing set for  $\gamma^{-1}\mathcal{L}$ . Now as  $[0, 1]$  is simply connected by [Lemma 2.2](#), we see that  $\gamma^{-1}\mathcal{L} = \underline{M}$  will be a constant sheaf. As  $[0, 1]$  is connected,  $\underline{M}([0, 1]) = M$ . It follows that the natural map  $\gamma^{-1}\mathcal{L}([0, 1]) = \underline{M}([0, 1]) \rightarrow \underline{M}_x = (\gamma^{-1}\mathcal{L})_x$  sending  $m$  to the germ  $(m, [0, 1])$  is an isomorphism for any  $x \in [0, 1]$ . Applying this to  $x = 0, 1$  we obtain a chain of explicit isomorphisms  $\gamma_* : L_{\gamma(0)} \rightarrow L_{\gamma(1)}$  called the monodromy map

$$\mathcal{L}_{\gamma(0)} \cong (\gamma^{-1}\mathcal{L})_0 \cong \gamma^{-1}\mathcal{L}(I) \cong (\gamma^{-1}\mathcal{L})_1 \cong \mathcal{L}_{\gamma(1)}$$

where the first isomorphism is the isomorphism of the stalk of the pullback sheaf, i.e. we send  $(s, U) \mapsto ((s, U), \gamma^{-1}(U))$  and similarly with the last isomorphism. It turns out that  $\gamma_*$  satisfy a bunch of nice properties, namely

**Lemma 2.3.** Suppose  $\mathcal{L} \in \mathrm{mod}(A_X)$ . Then the monodromy map  $\gamma_*$  is

1.  $A$ -linear:  $\gamma_*(v + aw) = \gamma_*(v) + a\gamma_*(w)$ .
2. homotopy invariant: if  $\gamma \sim \gamma'$ , then  $\gamma_* = \gamma'_*$ .
3. compatible with composition of paths:  $\gamma'_*(\gamma_*(x)) = (\gamma' \cdot \gamma)_*(x)$

This shows that the assignment of each point of  $x$  to it's stalk gives us a functor from the fundamental groupoid  $\Pi(X)$  to  $A$ -modules. By considering loops, composition of paths and homotopy invariance

gives us a group homomorphism  $\pi_1(X, x_0) \rightarrow \text{Aut}(\mathcal{L}_{x_0}) := \text{Aut}(M)$  and by  $A$ -linearity it will land in  $\text{Aut}_A(M)$ . This map

$$\pi_1(X, x_0) \rightarrow \text{Aut}_A(M)$$

is called the **monodromy representation** of  $\mathcal{L}$ . To go back consider the space  $(\tilde{X} \times M)/\pi_1(X, x_0)$  over  $X$  where  $\pi_1(X, x_0)$  acts on  $\tilde{X}$  by monodromy and  $M$  has the discrete topology. Take it's sheaf of sections to recover  $\mathcal{L}$ .

## 2.1 Cohomology of Local Systems

**Definition 2.4.** Let  $\mathcal{L}$  be an  $A$ -local system on  $X$ . Then the  $i$ -th cohomology group of  $\mathcal{L}$  will simply be the sheaf cohomology group

$$H^i(X, \mathcal{L}) = H_{A_X}^i(X, \mathcal{L})$$

i.e. pick an injective (acyclic) resolution of  $\mathcal{L}$  as an  $A_X$  module and then take cohomology after applying global sections.

**Remark.** Our definition above uses the definition of a local system as a sheaf but the other equivalent definitions will give rise to isomorphic cohomology theories. Covariant functors from the fundamental groupoid will give singular cohomology with local coefficients while representations of  $\pi_1(X, x_0)$  will give rise to a chain complex similar to group cohomology/equivariant cohomology (this is what's in Hatcher).

## 3 DeRham Cohomology

Recall that the slightly more general DeRham complex  $\text{DR}(X, V)$  of a smooth  $n$ -dimensional manifold  $X$  with values in an  $\mathbb{R}$  vector space  $V = \mathbb{R}^n$  is the complex

$$0 \rightarrow \Omega^0(X, V) \xrightarrow{d} \Omega^1(X, V) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(X, V) \rightarrow 0$$

where  $\Omega_X^k(X, V)$  =space of smooth differential  $k$ -forms taking values in  $V$ . This means if  $\Omega^k(X) = \bigwedge^k T^*X$ <sup>1</sup> and  $\overline{\mathbb{R}^n}$  is the trivial rank  $n$  vector bundle  $X \times \mathbb{R}^n \rightarrow X$ , then

$$\Omega_X^k(X, V) = \Gamma\left(X, \Omega^k(X) \otimes \overline{\mathbb{R}^n}\right) = \text{Hom}_{C^\infty(X)}\left(\bigwedge^k \text{Vect}(X), C^\infty(M)^n\right)$$

Note that the RHS means that we have an assignment at each point  $p \in X$  where we feed in  $k$  tangent vectors in  $T_p X$  and get back a vector in  $\mathbb{R}^n$  such that the assignment is smooth in  $p$ . The differential  $d: \Omega^k(X, V) \rightarrow \Omega^{k+1}(X, V)$  is given by

$$d\omega(X_0, \dots, X_k) = \sum (-1)^i X_i \cdot \omega(X_0, \dots, \widehat{X}_i, \dots, X_k) + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k)$$

where  $\omega \in \Omega^k(X)$  and  $\omega \otimes v \in \Omega^k(X, V)$  and  $X_0, \dots, X_k$  are vector fields. The DeRham cohomology of  $X$  with values in  $V$ , denoted by  $H^*(\Omega^*(X, V))$  will then be the cohomology of the above complex.

We can generalize above to define DeRham cohomology of  $X$  with values in a local system. However, we will need a remark that I previously said you shouldn't worry too much about, namely

<sup>1</sup>aka regular differential  $k$ -forms.

**Theorem 3.1.** *For  $X$  real smooth or complex analytic, we have an equivalence of categories between  $\mathbb{R}/\mathbb{C}$  local systems on  $X$  and vector bundles on  $X$  with a flat connection.*

*Proof.* We will just give the functors in both directions. Given a local system  $\mathcal{L}$ , consider the sheaf  $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{R}_X} \mathcal{O}_X$  where  $\mathcal{O}_X(U) = C^\infty(U, \mathbb{R})^2$  is the sheaf of smooth functions on  $X$ . Then on an trivializing cover  $\{U_i\}$  of  $\mathcal{L}$ , we have that  $\mathcal{L}|_{U_i}$  will be a constant sheaf and thus is of the form  $\underline{M}$  where  $M$  is a  $\mathbb{R}$ -module and thus is of the form  $\mathbb{R}^n$ . It follows that

$$\mathcal{V}|_{U_i} = (\mathcal{L} \otimes_{\mathbb{R}_X} \mathcal{O}_X)|_{U_i} \cong (\mathbb{R}_X^n \otimes_{\mathbb{R}_X} \mathcal{O}_X)|_{U_i} \cong \mathcal{O}_X^n$$

Thus  $\mathcal{V}$  is a vector bundle. Using the fact that  $\mathcal{L}$  is locally constant, we can then glue together the trivial flat connection on  $\mathcal{O}_X$  to obtain a flat connection  $\nabla_{\mathcal{L}}$  on  $\mathcal{V}$ . Conversely given a flat connection  $\nabla$ , take horizontal sections to get a local system like in the example above.

**Definition 3.2.** *Given a vector bundle  $E$  on  $X$ , we can define the differential forms with values in  $E$  as sections of*

$$\Omega^k(X, E) = \Gamma(X, \Omega^k(X) \otimes E)$$

**Definition 3.3.** *A connection on a (real) vector bundle  $E$  on a manifold  $M$  is a  $\mathbb{R}$ -linear map*

$$\nabla : \Gamma(M, E) \rightarrow \Omega^1(M, E)$$

*satisfying the Leibniz rule  $\nabla(fs) = df \otimes s + f\nabla(s)$ .*

When  $\nabla$  is a flat connection on  $E$  we can then form the twisted DeRham complex

$$\text{DR}(X)_E^\nabla : 0 \rightarrow \Omega^0(X, E) \xrightarrow{\nabla} \Omega^1(X, E) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^n(X, E) \rightarrow 0$$

where the differential  $\nabla$  is given by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|} \omega \wedge \nabla(s)$$

The fact that  $\nabla$  is flat is the statement that  $\nabla \circ \nabla = 0$ , i.e. that we actually get a complex. Observe that for  $E = \mathcal{O}_X$  the trivial bundle and  $\nabla$  the trivial connection this returns the regular DeRham complex.

**Remark.** One can alternatively define the DeRham complex for vector bundles with flat connections (aka local systems)  $\mathcal{L}$  like so. Let  $\mathcal{V}_{\mathcal{L}} = \mathcal{L} \otimes_{\mathbb{R}_X} \mathcal{O}_X$  be the vector bundle associated to  $\mathcal{L}$ . Define the twisted DeRham complex to be

$$\Omega^k(X, \mathcal{L}) := \Omega^k(X) \otimes_{\mathcal{O}_X} \mathcal{V}_{\mathcal{L}} = \Omega^k(X) \otimes_{\mathcal{O}_X} (\mathcal{L} \otimes_{\mathbb{R}_X} \mathcal{O}_X) = \Omega^k(X) \otimes_{\mathbb{R}_X} \mathcal{L}$$

Then let  $\{U_i\}_{i \in I}$  be a connected cover which trivializes both  $\Omega^k(X)$  and  $\mathcal{L}$ . Given  $\omega \otimes s \in \Omega^k(X, \mathcal{L}) = \Omega^k(X) \otimes_{\mathbb{R}_X} \mathcal{L}$ , by choosing a basis of local sections, we have  $(\omega \otimes s)|_{U_i} = \omega_i \otimes s_i \in \Omega^k(U_i, \mathcal{L})$ . Then we claim the local differential defined by

$$d|_{U_i} : \Omega^k(U_i, \mathcal{L}) \rightarrow \Omega^{k+1}(U_i, \mathcal{L}), \quad \omega_i \otimes s_i \mapsto d\omega_i \otimes s_i$$

is well defined, aka is  $\mathbb{R}_X$ -linear. But since  $U_i$  is connected,  $\mathbb{R}_X(U_i) = \mathbb{R}$  and  $d$  is clearly  $\mathbb{R}$  linear.<sup>3</sup> One then checks that the set of  $\{d\omega\}_{i \in I}$  are compatible and thus glue back up to a global section. Moreover as locally  $d^2|_{U_i} = 0$ ,  $d^2 = 0$  globally as well so we do have a complex.

<sup>2</sup>Aka  $\mathcal{O}_X$  is the trivial bundle on  $X$ .

<sup>3</sup>Note if we had tried to show it was  $\mathcal{O}_X$ -linear then  $d(f\omega) = df \wedge \omega + f d\omega$  which is a problem and thus why we need  $\nabla$  to be the differential instead.

**Remark.** Letting  $E = \mathbb{R}^n \times X \rightarrow X$  be the trivial rank  $n$  vector bundle with the trivial connection  $\nabla = d$ , we recover  $\Omega(X, V)$  as defined originally.

**Definition 3.4.** Given a local system  $\mathcal{L}$  on a real smooth manifold  $M$ , the twisted deRham cohomology of  $\mathcal{L}$  is defined to be

$$H_{dR}^*(M, \mathcal{L}) = H^*(\Omega^*(M; \mathcal{V}, \nabla_{\mathcal{L}}))$$

where  $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{R}_X} \mathcal{O}_X$  as in above.

**Theorem 3.5** (Twisted de Rham). Let  $\mathcal{L}$  be a local system on a smooth real manifold  $M$ . Then there is an isomorphism

$$H^k(M, \mathcal{L}) \cong H_{dR}^k(M, \mathcal{L})$$

**Remark.** For complex analytic, one needs to replace the RHS with hypercohomology.

## 4 Main Result for $X_\Gamma$

All representations are over  $\mathbb{R}$  or  $\mathbb{C}$  and likewise any local system will be in  $\mathbb{R}_X$  or  $\mathbb{C}_X$  where  $X = G/K$  where  $K$  is a maximal compact subgroup of  $G$ . Let  $X_\Gamma = \Gamma \backslash X$  be the coset/orbit space of  $\Gamma \subseteq G$  acting on  $X$ . We first need

**Theorem 4.1** (Cartan, Iwasawa, Malcev). Any connected Lie group  $G$  has a maximal compact subgroup  $K$ . All maximal compact subgroups are conjugate to each other and moreover  $G/K$  is diffeomorphic to Euclidean space.

This is actually true without the connectedness assumption but the person who proved it isn't as recognizable, so we do not state it. Now consider the fiber bundle  $\Gamma \rightarrow X \rightarrow X_\Gamma$ . By the long exact sequence in homotopy groups

$$\dots \rightarrow \pi_n(X) \rightarrow \pi_n(X_\Gamma) \rightarrow \pi_{n-1}(\Gamma) \rightarrow \dots \rightarrow \pi_1(X) \rightarrow \pi(X_\Gamma) \rightarrow \pi_0(\Gamma) \rightarrow \pi_0(X) \rightarrow \pi_0(X_\Gamma) \rightarrow 0$$

$X$  is diffeomorphic to Euclidean space, so is contractible and thus  $\pi_k(X) = 0 \forall k$ ,  $\Gamma$  being discrete has the same fate except  $\pi_0(\Gamma) = \Gamma$  from which it follows that  $\pi_1(X_\Gamma) = \Gamma$  and all other homotopy groups of  $X_\Gamma$  vanish. Thus, we see that  $X_\Gamma$  is a  $K(\Gamma, 1)$ . Also,  $\Gamma$  being discrete again means that  $X \rightarrow X_\Gamma$  is a covering map and so since  $X$  is simply connected, it's the universal cover for  $X_\Gamma$ .

Now given a representation  $V$  of  $\Gamma$ , since  $\pi_1(X_\Gamma) = \Gamma$  recall that we then get an associated local system  $\tilde{V}$  on  $X_\Gamma$  as the sheaf of sections of  $X \times_\Gamma V \rightarrow X_\Gamma$  since  $X$  is the universal cover. But since  $X_\Gamma$  is a  $K(\Gamma, 1)$  we also have

$$H^k(\Gamma, V) = H^k(K(\Gamma, 1), \tilde{V}) = H^k(X_\Gamma, \tilde{V}) \quad (1)$$

because the complex corresponding to the standard bar resolution of the trivial module  $\mathbb{Z}$

$$\dots \rightarrow (\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \rightarrow \dots \rightarrow \mathbb{Z} \rightarrow 0$$

coincides with the complex of singular/simplicial chains of the geometric realization of the simplicial set

$$\dots \rightrightarrows G \times G \times G \rightrightarrows G \times G \rightarrow G$$

gotten by the bar construction which turns out to be a  $K(G, 1)$ . Applying  $\text{Hom}_{\mathbb{Z}[G]}(-, V)$  will correspond to twisting the singular chains by  $\tilde{V}$  and since the complexes are the same we get the desired

isomorphism of group cohomology of  $\Gamma$  with singular cohomology of  $K(\Gamma, 1)$ .

The reason why we defined deRham cohomology with values in a vector space is that groups can now act on the cochains. Explicitly,

**Definition 4.2.** *Given a group  $G$  that acts on a manifold  $M$  by diffeomorphisms and a linear representation  $(V, \rho)$  of  $G$ , then  $G$  acts on  $\Omega^k(M, V)$  by*

$$(g \cdot \omega)_p(X_1, \dots, X_k) = \rho(g)\omega_{g^{-1} \cdot p}(g^{-1} \cdot X_1, \dots, g^{-1} \cdot X_k)$$

*Alternatively, since  $\Omega^k(M, V) = \Gamma(M, \Omega^k(X) \otimes \bar{V})$ , this is equivalent to the map sending  $\omega \otimes v \mapsto (g^{-1})^*(\omega) \otimes g \cdot v$ .*

### Theorem 2

*Let  $\Gamma$  be a discrete subgroup of a Lie Group  $G$  with  $X = G/K$  and  $X_\Gamma = \Gamma \backslash X$  and let  $V$  be a real representation of  $\Gamma$ . Then there is a canonical isomorphism*

$$H^*(\Gamma, V) \cong H^*(\Omega^*(X, V)^\Gamma)$$

*Proof.* Case 1:  $\Gamma$  acts freely on  $X$  (E.g. when  $\Gamma$  is torsion free as stabilizers must be finite by the proper action of  $\Gamma$  but torsion free implies they are all trivial) In this case  $X_\Gamma = \Gamma \backslash X$  will be a manifold being the quotient of a free and proper action. Thus we can apply [Eq. \(1\)](#) and [Theorem 3.5](#) to obtain

$$H^*(\Gamma, V) \cong H^k(X_\Gamma, \tilde{V}) \cong H_{dR}^k(X_\Gamma, \tilde{V})$$

Consider the map  $\pi : X \rightarrow X_\Gamma$ . We then have a map on forms given by

$$\begin{aligned} \pi^* : \Omega^k(X_\Gamma, \tilde{V}) &\rightarrow \Omega^k(X, \pi^*(\tilde{V})) \rightarrow \Omega^k(X, V)^\Gamma \\ \omega \otimes s &\mapsto \pi^*(\omega) \otimes \pi^*(s) \end{aligned}$$

where we claim that  $\Omega^k(X, \pi^*(\tilde{V})) = \Omega^k(X, V)$  and that the image of the map lands in the  $\Gamma$  invariant forms. Recall that  $X$  is the universal cover for  $X_\Gamma$  and so  $\tilde{V}$  is defined to be the sheaf of sections of  $\tilde{\pi} : X \times_\Gamma V \rightarrow X_\Gamma$  where the map is just  $\pi$  in the first component. Pulling this back along  $\pi$  which is essentially the same map, it follows that the pullback bundle will just be  $X \times V$  by the definition of pullback. But the sheaf of sections of this will be the trivial rank  $n$  local system and thus when tensoring up we get the trivial rank  $n$  vector bundle and thus we have  $\Omega^k(X, V)$ . Since  $\pi$  is seen to be  $\Gamma$  equivariant we have

$$\gamma \cdot (\pi^*(\omega \otimes s)) = \gamma \cdot (\pi^*\omega \otimes \pi^*s) = (\gamma^{-1})^*\pi^*\omega \otimes \gamma\pi^*s = \pi^*(\gamma^{-1})^*\omega \otimes \pi^*\gamma s = \pi^*\omega \otimes \pi^*s$$

Moreover, this map is an isomorphism because any  $\Gamma$  invariant form in  $\Omega^k(X, V)^\Gamma$  will descend to the quotient  $X_\Gamma$  and one can check these two maps are inverses to each other. Since the complexes are isomorphic, their cohomologies will be the same and so we get that

$$H_{dR}^k(X_\Gamma, \tilde{V}) \cong H^*(\Omega^*(X, V)^\Gamma)$$

Case 2:  $\Gamma$  has a torsion free normal subgroup of finite index. Let  $H \subseteq \Gamma$  be the torsion free normal subgroup of finite index. Note that  $\Gamma/H$  acts on  $H^*(H, V)$  by  $(g \cdot f)(h_1, \dots) = g \cdot f(g^{-1}h_1g, \dots)$ . By the Hochschild-Serre spectral sequence we have that

$$H^p(\Gamma/H, H^q(H, V)) \implies H^{p+q}(\Gamma, V)$$

Notice that  $H^q(H, V)$  is a cohomologically trivial  $\Gamma/H$  module because  $H^q(H, V)$  is a vector space and  $\Gamma/H$  is finite and so higher cohomology dies for  $p \geq 1$  and thus we have the equality

$$H^*(\Gamma, V) = H^*(H, V)^{\Gamma/H}$$

On the other hand, as  $(-)^{\Gamma} = \left( (-)^H \right)^{\Gamma/H}$ , we have

$$H^*(\Omega^*(X, V)^{\Gamma}) = H^*\left(\left(\Omega^*(X, V)^H\right)^{\Gamma/H}\right) = H^*(\Omega^*(X, V)^H)^{\Gamma/H}$$

where the last equality is because taking invariants of a finite group on a vector space is exact in characteristic 0 and thus we can take invariants after taking cohomology. But now we can use Case 1 and the proof is finished. Case 2 is sufficient for our purposes when  $\Gamma$  is an arithmetic subgroup.